

Chapter Six:

Steady State Power Analysis

1. Instantaneous power

The power $p(t)$ absorbed by an element is the product of voltage $v(t)$ across the element and current $i(t)$ through it.

$$p(t) = v(t)i(t)$$

The instantaneous power (in watts) is the power at any instant of time. It is the rate at which an element absorbs energy.

$$v(t) = V_m \cos(\omega t + \theta_v)$$

$$i(t) = I_m \cos(\omega t + \theta_i)$$

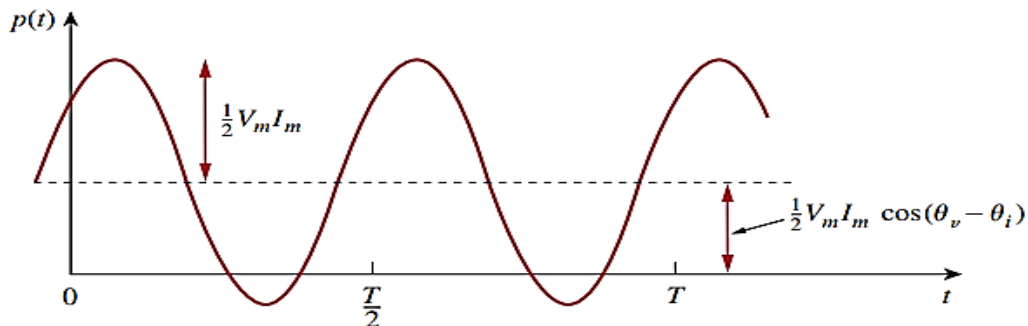
$$p(t) = v(t)i(t)$$

$$= V_m I_m \cos(\omega t + \theta_v) \cos(\omega t + \theta_i)$$

From trigonometric identity $\cos A \cos B = \frac{1}{2}[\cos(A - B) + \cos(A + B)]$

$$p(t) = \frac{1}{2} V_m I_m \cos(\theta_v - \theta_i) + \frac{1}{2} V_m I_m \cos(2\omega t + \theta_v + \theta_i)$$

- This shows us that the instantaneous power has two parts.
 - i. The first part is constant or time independent. Its value depends on the phase difference between the voltage and the current.
 - ii. The second part is a sinusoidal function whose frequency is which is twice the angular frequency of the voltage or current.



We observe that $p(t)$ is positive for some part of each cycle and negative for the rest of the cycle.

- ❖ When $p(t)$ is positive, power is absorbed by the circuit.
- ❖ When $p(t)$ is negative, power is absorbed by the source; that is, power is transferred from the circuit to the source. This is possible because of the storage elements (capacitors and inductors) in the circuit.

2. Average power,

is the average of the instantaneous power over one period. Thus, the average power is given by

$$\begin{aligned} P &= \frac{1}{T} \int_0^T p(t) dt & P &= \frac{1}{T} \int_0^T \frac{1}{2} V_m I_m \cos(\theta_v - \theta_i) dt + \frac{1}{T} \int_0^T \frac{1}{2} V_m I_m \cos(2\omega t + \theta_v + \theta_i) dt \\ & & &= \frac{1}{2} V_m I_m \cos(\theta_v - \theta_i) \frac{1}{T} \int_0^T dt + \frac{1}{2} V_m I_m \frac{1}{T} \int_0^T \cos(2\omega t + \theta_v + \theta_i) dt \end{aligned}$$

The first integrand is constant, and the average of a constant is the same constant. The second integrand is a sinusoid. We know that the average of a sinusoid over its period is zero because the area under the sinusoid during a positive half-cycle is canceled by the area under it during the following negative half-cycle.

$$P = \frac{1}{2} V_m I_m \cos(\theta_v - \theta_i)$$

To use phasors, we notice that

$$\frac{1}{2} \mathbf{V} \mathbf{I}^* = \frac{1}{2} V_m I_m \angle \theta_v - \theta_i = \frac{1}{2} V_m I_m [\cos(\theta_v - \theta_i) + j \sin(\theta_v - \theta_i)]$$

$$P = \frac{1}{2} \text{Re}[\mathbf{V} \mathbf{I}^*] = \frac{1}{2} V_m I_m \cos(\theta_v - \theta_i)$$

When $\theta_v = \theta_i$, the voltage and current are in phase. This implies a purely resistive circuit or resistive load R , and

$$P = \frac{1}{2} V_m I_m = \frac{1}{2} I_m^2 R = \frac{1}{2} |\mathbf{I}|^2 R$$

When $\theta_v - \theta_i = \pm 90^\circ$, we have a purely reactive circuit, and $P = \frac{1}{2} V_m I_m \cos 90^\circ = 0$

- A resistive load (R) absorbs power at all times, while a reactive load (L or C) absorbs zero average power.

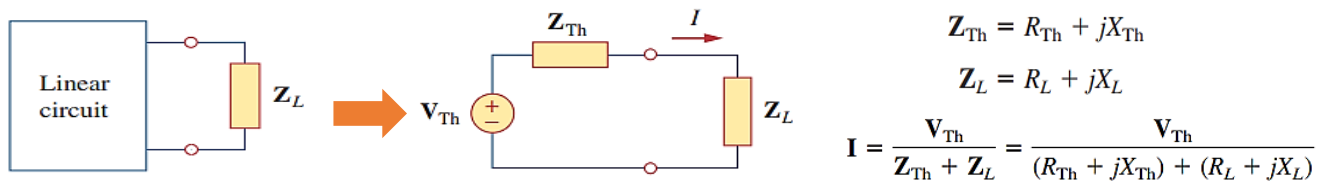
Example 1: Given that $v(t) = 120 \cos(377t + 45^\circ)$ V and $i(t) = 10 \cos(377t - 10^\circ)$ A find the instantaneous power and the average power absorbed by the passive linear network.

Example 2: Calculate the average power absorbed by an impedance $\mathbf{Z} = 30 - j70 \Omega$ when a voltage $\mathbf{V} = 120 \angle 0^\circ$ is applied across it.

Maximum Average Power Transfer

In DC circuit analysis we solved the problem of maximizing the power delivered by a power-supplying resistive network to a load. This is done by representing the circuit by its Thevenin equivalent, we proved that the maximum power would be delivered to the load if the load resistance is equal to the Thevenin resistance.

We now extend that result to ac circuits.



$$P = \frac{1}{2} |\mathbf{I}|^2 R_L = \frac{|\mathbf{V}_{Th}|^2 R_L / 2}{(R_{Th} + R_L)^2 + (X_{Th} + X_L)^2}$$

$$\frac{\partial P}{\partial X_L} = - \frac{|\mathbf{V}_{Th}|^2 R_L (X_{Th} + X_L)}{[(R_{Th} + R_L)^2 + (X_{Th} + X_L)^2]^2}$$

$$\frac{\partial P}{\partial R_L} = \frac{|\mathbf{V}_{Th}|^2 [(R_{Th} + R_L)^2 + (X_{Th} + X_L)^2 - 2R_L(R_{Th} + R_L)]}{2[(R_{Th} + R_L)^2 + (X_{Th} + X_L)^2]^2}$$

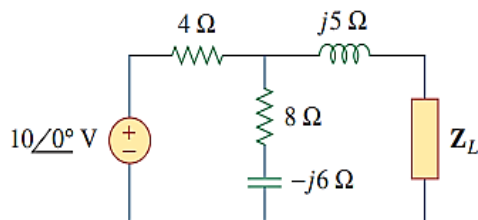
$$\mathbf{Z}_L = R_L + jX_L = R_{Th} - jX_{Th} = \mathbf{Z}_{Th}^*$$

- For maximum average power transfer, the load impedance \mathbf{Z}_L must be equal to the complex conjugate of the Thevenin impedance \mathbf{Z}_{Th} .

The maximum transfer power is given by

$$P_{\max} = \frac{|\mathbf{V}_{Th}|^2}{8R_{Th}}$$

Example 3: Determine the load impedance that maximizes the average power drawn from the circuit shown below. What is the maximum average power? **Ans $P_{\max} = 39.29 \text{ W}$**



3. EFFECTIVE OR RMS VALUE

The idea of *effective value* arises from the need to measure the effectiveness of a voltage or current source in delivering power to a resistive load.

The **effective value** of a periodic current is the dc current that delivers the same average power to a resistor as the periodic current.

The average power absorbed by the resistor in the ac circuit is

$$P = \frac{1}{T} \int_0^T i^2 R dt = \frac{R}{T} \int_0^T i^2 dt$$

power absorbed by the resistor in the dc circuit is $P = I_{\text{eff}}^2 R$

from the above two equations

$$I_{\text{eff}} = \sqrt{\frac{1}{T} \int_0^T i^2 dt}$$

For the sinusoid $i(t) = I_m \cos \omega t$, the effective or rms value is

$$I_{\text{rms}} = \sqrt{\frac{1}{T} \int_0^T I_m^2 \cos^2 \omega t dt} = \sqrt{\frac{I_m^2}{T} \int_0^T \frac{1}{2} (1 + \cos 2\omega t) dt} = \frac{I_m}{\sqrt{2}}$$

Similarly, $v(t) = V_m \cos \omega t$, $V_{\text{rms}} = V_m / \sqrt{2}$

The average power can be written in terms of the rms values.

$$\begin{aligned} P &= \frac{1}{2} V_m I_m \cos(\theta_v - \theta_i) = \frac{V_m}{\sqrt{2}} \frac{I_m}{\sqrt{2}} \cos(\theta_v - \theta_i) \\ &= V_{\text{rms}} I_{\text{rms}} \cos(\theta_v - \theta_i) \end{aligned}$$

APPARENT POWER AND POWER FACTOR

if the voltage and current at the terminals of a circuit are $v(t) = V_m \cos(\omega t + \theta_v)$ and $i(t) = I_m \cos(\omega t + \theta_i)$, the average power is

$$P = \frac{1}{2} (V_m I_m \cos(\theta_v - \theta_i)),$$

$$P = V_{\text{rms}} I_{\text{rms}} \cos(\theta_v - \theta_i) = S \cos(\theta_v - \theta_i)$$

The average power is a product of two terms. The product V_{rms} and I_{rms} is known as the *apparent power* S . The factor $\cos(\theta_v - \theta_i)$ is called the *power factor* (pf).

$$P = V_{\text{rms}} I_{\text{rms}} \cos(\theta_v - \theta_i) = S \cos(\theta_v - \theta_i)$$

The power factor is dimensionless, since it is the ratio of the average power to the apparent,

$$\text{pf} = \frac{P}{S} = \cos(\theta_v - \theta_i)$$

The angle $\theta_v - \theta_i$ is called the *power factor angle*, since it is the angle, whose cosine is the power factor. The power factor angle is equal to the angle of the load impedance if \mathbf{V} is the voltage across the load and \mathbf{I} is the current through it. This is evident from the fact that

$$\mathbf{Z} = \frac{\mathbf{V}}{\mathbf{I}} = \frac{V_m \angle \theta_v}{I_m \angle \theta_i} = \frac{V_m}{I_m} \angle \theta_v - \theta_i = \frac{V_{\text{rms}}}{I_{\text{rms}}} \angle \theta_v - \theta_i$$

The **power factor** is the cosine of the phase difference between voltage and current. It is also the cosine of the angle of the load impedance

The power factor may be seen as that factor by which the apparent power must be multiplied to obtain the real or average power. The value of pf ranges between zero and unity.

For a purely resistive load, the voltage and current are in phase, so that $\theta_v - \theta_i = 0$ and $\text{pf} = 1$.

This implies that the apparent power is equal to the average power.

For a purely reactive load, $\theta_v - \theta_i = \pm 90^\circ$ and $\text{pf} = 0$. In this case the average power is zero.

In between these two extreme cases, pf is said to be *leading* or *lagging*. Leading power factor means that current leads voltage, which implies a capacitive load. Lagging power factor means that current lags voltage, implying an inductive load. Power factor affects the electric bills consumers pay the electric utility companies,

Example 4: A series-connected load draws a current $i(t) = 4 \cos(100\pi t + 10^\circ)$ A when the applied voltage is $v(t) = 120 \cos(100\pi t - 20^\circ)$ V. Find the apparent power and the power factor of the load. Determine the element values that form the series-connected load.

COMPLEX POWER

The *complex power* \mathbf{S} absorbed by the ac load is the product of the voltage and the complexconjugate of the current, or

$$\mathbf{S} = \frac{1}{2} \mathbf{V} \mathbf{I}^* = \mathbf{V}_{\text{rms}} \mathbf{I}_{\text{rms}}^*$$

Where,

$$\mathbf{V}_{\text{rms}} = \frac{\mathbf{V}}{\sqrt{2}} = V_{\text{rms}} \angle \theta_v \quad \mathbf{I}_{\text{rms}} = \frac{\mathbf{I}}{\sqrt{2}} = I_{\text{rms}} \angle \theta_i$$

$$\begin{aligned} \mathbf{S} &= V_{\text{rms}} I_{\text{rms}} \angle \theta_v - \theta_i \\ &= V_{\text{rms}} I_{\text{rms}} \cos(\theta_v - \theta_i) + j V_{\text{rms}} I_{\text{rms}} \sin(\theta_v - \theta_i) \end{aligned}$$

The complex power may be expressed in terms of the load impedance \mathbf{Z} .

$$\mathbf{Z} = \frac{\mathbf{V}}{\mathbf{I}} = \frac{\mathbf{V}_{\text{rms}}}{\mathbf{I}_{\text{rms}}} = \frac{V_{\text{rms}}}{I_{\text{rms}}} \angle \theta_v - \theta_i, \quad \mathbf{V}_{\text{rms}} = \mathbf{Z} \mathbf{I}_{\text{rms}}.$$

Since $\mathbf{Z} = R + jX$

$$\mathbf{S} = I_{\text{rms}}^2 (R + jX) = P + jQ$$

where P and Q are the real and imaginary parts of the complex power; that is,

$$\begin{aligned} P &= \text{Re}(\mathbf{S}) = I_{\text{rms}}^2 R \\ Q &= \text{Im}(\mathbf{S}) = I_{\text{rms}}^2 X \end{aligned}$$

P is the average or real power and it depends on the load's resistance R . Q depends on the load's reactance X and is called the *reactive* (or quadrature) power.

$$P = V_{\text{rms}} I_{\text{rms}} \cos(\theta_v - \theta_i), \quad Q = V_{\text{rms}} I_{\text{rms}} \sin(\theta_v - \theta_i)$$

The real power P is the average power in watts delivered to a load; it is the only useful power. It is the actual power dissipated by the load.

The reactive power Q is a measure of the energy exchange between the source and the reactive part of the load. The unit of Q is the *volt-ampere reactive* (VAR) to distinguish it from the real power, whose unit is the watt.

Energy storage elements neither dissipate nor supply power, but exchange power back and forth with the rest of the network. In the same way, the reactive power is being transferred back and forth between the load and the source. It represents a lossless interchange between the load and the source.

- Notice that:
 1. $Q=0$ for resistive loads (unity pf).
 2. $Q<0$ for capacitive loads (leading pf).
 3. $Q>0$ for inductive loads (lagging pf).
- Complex power (in VA) is the product of the rms voltage phasor and the complex conjugate of the rms current phasor. As a complex quantity, its real part is real power P and its imaginary part is reactive power Q .

$$\text{Complex Power} = \mathbf{S} = P + jQ = \mathbf{V}_{\text{rms}}(\mathbf{I}_{\text{rms}})^*$$

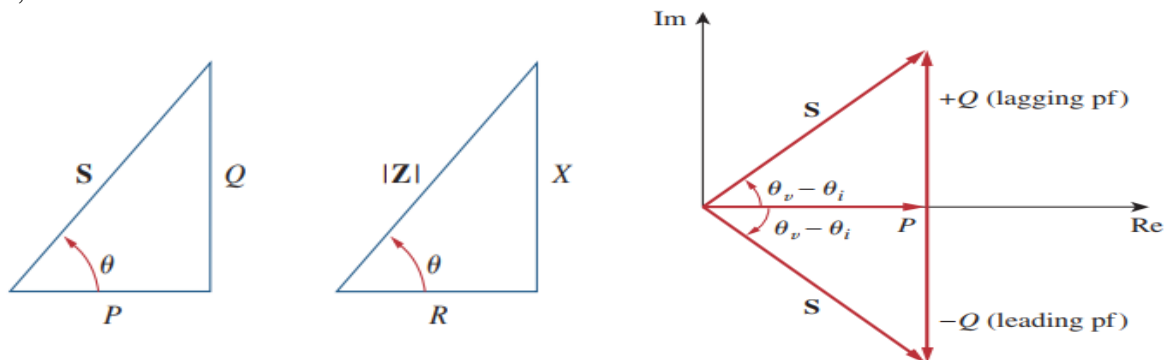
$$= |\mathbf{V}_{\text{rms}}| |\mathbf{I}_{\text{rms}}| \angle \theta_v - \theta_i$$

$$\text{Apparent Power} = S = |\mathbf{S}| = |\mathbf{V}_{\text{rms}}| |\mathbf{I}_{\text{rms}}| = \sqrt{P^2 + Q^2}$$

$$\text{Real Power} = P = \text{Re}(\mathbf{S}) = S \cos(\theta_v - \theta_i)$$

$$\text{Reactive Power} = Q = \text{Im}(\mathbf{S}) = S \sin(\theta_v - \theta_i)$$

It is a standard practice to represent \mathbf{S} , P , and Q in the form of a triangle, known as the power triangle,



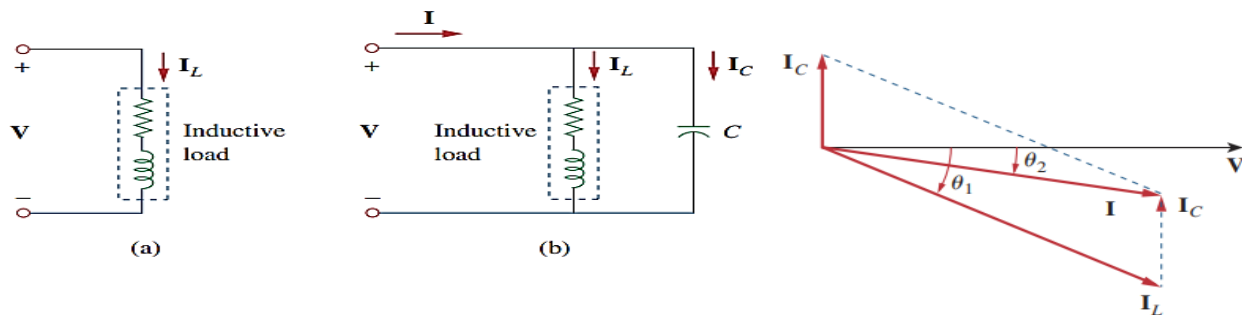
Example 5: The voltage across a load is $v(t) = 60 \cos(\omega t - 10^\circ)$ V and the current through the element in the direction of the voltage drop is $i(t) = 1.5 \cos(\omega t + 50^\circ)$ A. Find: (a) the complex and apparent powers, (b) the real and reactive powers, and (c) the power factor and the load impedance.

ANS: a). 45 (60°) VA b). $P = 22.5$ W, $Q = -38.97$ VAR c). 0.5 (leading)

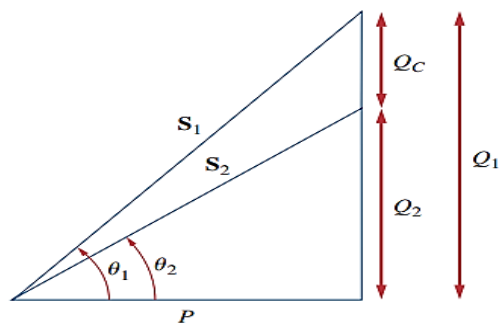
Power factor and power factor correction

The process of increasing the power factor without altering the voltage or current to the original load is known as power factor correction.

Since most loads are inductive, as shown in Fig.(a), a load's power factor is improved or corrected by deliberately installing a capacitor in parallel with the load, as shown in Fig. (b).



We can look at the power factor correction from another perspective. Consider the power triangle



$$P = S_1 \cos \theta_1, \quad Q_1 = S_1 \sin \theta_1 = P \tan \theta_1$$

If we desire to increase the power factor from $\cos \theta_1$ to $\cos \theta_2$ without altering the real power (i.e. $P = S_2 \cos \theta_2$), then the new reactive power is

$$Q_2 = P \tan \theta_2$$

- The reduction in the reactive power is caused by the shunt capacitor; that is,

$$Q_C = Q_1 - Q_2 = P(\tan \theta_1 - \tan \theta_2) \quad Q_C = V_{\text{rms}}^2 / X_C = \omega C V_{\text{rms}}^2$$

- The value of the required shunt capacitance C is determined as

$$C = \frac{Q_C}{\omega V_{\text{rms}}^2} = \frac{P(\tan \theta_1 - \tan \theta_2)}{\omega V_{\text{rms}}^2}$$

Note that the real power P dissipated by the load is not affected by the power factor correction because the average power due to the capacitance is zero.

Although the most common situation in practice is that of an inductive load, it is also possible that the load is capacitive, that is, the load is operating at a leading power factor. In this case, an inductor should be connected across the load for power factor correction. The required shunt inductance L can be calculated from

$$Q_L = \frac{V_{\text{rms}}^2}{X_L} = \frac{V_{\text{rms}}^2}{\omega L} \quad \Rightarrow \quad L = \frac{V_{\text{rms}}^2}{\omega Q_L}$$

where $Q_L = Q_1 - Q_2$, the difference between the new and old reactive powers.

Example 6 : When connected to a 120-V (rms), 60-Hz power line, a load absorbs 4 kW at a lagging power factor of 0.8. Find the value of capacitance necessary to raise the pf to 0.95.

Example 7: Find the value of parallel capacitance needed to correct a load of 140 kVAR at 0.85 lagging pf to unity pf. Assume that the load is supplied by a 110-V (rms), 60-Hz line.